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Preliminary Algebraic Geometry.

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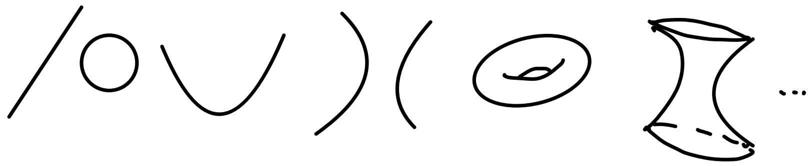
- 助教自我介绍
- 交作业与发作业时间：周二上课前后.

平时成绩 (100') p
期末成绩 (100') q
最终成绩: $\max\{q, 0.2\times p+0.8\times q\}$

e.g.



Algebraic geometry = studying zeros of multivariate polynomials by using of abstract algebraic technique.

fundamental objects = algebraic varieties

aim of this course: study algebraic curves.

§1.1 algebraic preliminaries.

Ring means a commutative ring with multiplicative identity. (in this course)

domain = ring + cancellation law (= without zero divisors)

field = ring + every nonzero element is a unit.

e.g. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_p$, $\mathbb{Z}[x]$, ...

ring homomorphism = map preserves $+, \cdot, 1$.

ideals and quotient ring (residue class ring)

$I \lhd R \Rightarrow$ ring homo. $R \twoheadrightarrow R/I$

Fact: (1). $\mathrm{Hom}_{ring}(R/I, S) \xrightarrow{1:1} \left\{\varphi : R \to S \;\middle|\; \begin{array}{l} \varphi = \text{ring hom,} \\ \varphi(I) = 0 \end{array}\right\}$.

(2). I = prime $\Leftrightarrow$ R/I = domain

(3). I = maximal $\Leftrightarrow$ R/I = field

(4). maximal $\Rightarrow$ prime.

Characteristic. Let R be a ring

$$\mathrm{Char}\, R := \begin{cases} \min \Sigma & \Sigma := \{n \in \mathbb{Z}_{\geq 1} \mid \overbrace{1+1+\cdots+1}^{n个} = 0 \text{ in } R\} \neq \phi \\ 0 & \Sigma = \phi \end{cases}$$

e.g. $\mathrm{char}(\mathbb{Z}/n\mathbb{Z}) = n$, $\mathrm{char}(\mathbb{Z}) = 0$, $\mathrm{char}(\mathbb{R}) = 0 = \mathrm{char}(\mathbb{C})$.

② Fact: R = domain $\Rightarrow$ $\mathrm{char}\, R = 0$ or $\mathrm{char}\, R$ = prime number

从已有的环构造新环:

quotient field K of a domain R:
or, field of fractions

$$K = \{(a,b) \mid a \in R\setminus\{0\},\ b \in R\}/\sim \qquad (a,b)\sim(c,d) \Leftrightarrow ad=bc$$

Fact: let L be a field, then

the set of injective ring hom.

$$\mathrm{Hom}_{\mathrm{ring}}(K, L) \xrightarrow{1:1} \mathrm{Hom}^{\mathrm{inj}}_{\mathrm{ring}}(R, L)$$

i.e. every injective ring homomorphism $R \hookrightarrow L$ extends uniquely ~~to~~ as a ring hom. $K \hookrightarrow L$.

$R[x_1, \cdots, x_n]$ = ring of polynomials in n variables over a ring R

monomial $x^{i} = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$
degree $|i| := i_1 + i_2 + \cdots + i_n$.

- We call $F = \sum_i a_i x^{i} \in R[x_1, \cdots, x_n]$ **homogeneous** or **a form** of degree d if $a_i = 0 \ \forall i : |i| \neq d$.

  ↳ a function defined on a vector space, which may be expressed as a homog. function of the coordinates over any basis

- $\forall F \Rightarrow F = F_0 + F_1 + F_2 + \cdots + F_d$ where F_i is a form of degree i.

  constant term → F_0, linear term → F_1, quadratic term → F_2

  $\cdot \deg F := d$

Fact: let $\varphi : R \to S$ be a ring homomorphism. Then

$$\begin{aligned} \mathrm{Hom}_{R\text{-alg}}(R[x_1, \cdots, x_n], S) &\xrightarrow{1:1} S^{\oplus n} \\ \tilde{\varphi} &\longmapsto (\tilde{\varphi}(x_1), \cdots, \tilde{\varphi}(x_n)) \end{aligned}$$

$$\{\tilde{\varphi} \in \mathrm{Hom}_{\mathrm{ring}}(R[x_1 \cdots x_n], S) \mid \tilde{\varphi}|_R = \varphi\}$$

e.g. $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1, \cdots, x_n], \mathbb{C}) \xrightarrow{1:1} \mathbb{C}^n$.

algebraically closed field.

R = ring, $a \in R$ & $F \in R[x]$. Then

$$F(a) = 0 \Rightarrow \exists!\, G \in R[x] \text{ s.t. } F = (x-a)G.$$

Def: A field is algebraically closed if any nonconstant $F \in k[x]$ has a root.

Fact: Let $k = \bar{k}$ be an algebraically closed field. Then

(i). $F = \mu \prod_i (x - \lambda_i)^{e_i}$ $\quad \mu \in k^*$, $\lambda_i \in k$ distinct roots of F

(ii). $\deg F = \sum_i e_i$

Example: $\mathbb{C}$ is algebraically closed.

Fact: Any alg. closed field is infinite.